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Anomalous dimensions in the Thirring model

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The dimensions of a self interacting spinor field in the Thirring model is calculated exactly using formal methods of quantum field theory. It is shown that there are two categories of solutions, scattering type and stationary type for the model and one of them agrees with the results of Johnson. It is also found that for a possible renormalized version of the theory, the dimension may become complex leading to non-hermitian Green's functions.

1. INTRODUCTION

One of the pioneering ideas of recent times is that the dimension of a field is influenced by the presence of forces. This concept has far reaching consequences in determing short distance behaviour of strong interactions, scaling behaviour and phase transitions. The only relativistic model where the idea has been exactly demonstrated is the soluble Thirring (1958) model. This is a two dimensional (one space and one time) quantum field theory of a massless relativistic spinor field interaction with itself. Using solutions of the model found by Johnson (1961), Wilson (1970) showed that the dimension of the spinor field changes with interaction reinforcing the hypothesis that dimensions of fields are dynamical quantities.

Originally Thirring (1958) had proposed this model to study the mathematical structure of relativistic quantum fields and to lay solid foundations for the conjectures contained in scattering theory like the existence of in and out fields, renormalization constants and the scattering matrix. Even though he was able to calculate one particle and 3-particles production amplitudes of the field operator by solving the Schrödinger equation, he was not able to solve other physical quantities of the model exactly. He had hoped that the model could be completely solved. Glaser (1958) initiated the use of methods of formal field theory and claimed to have obtained exact solutions to the model. In solving the model, he had used the fact that charge densities commute. So some of the solutions obtained by him were challenged by Pradhan (1958) who pointed out that the vacuum expectation values of commutators of the charge densities at unequal times is not zero but a derivative of the Dirac delta function anticipating the

Schwinger term (Pradhan 1958). Pradhan however, did not obtain a generally correct expression for the anti-ordered expression for a Moller-like matrix. This was pointed out by Scarf (1959) who insisted on the correctness of Glaser's work and made some progress in improving the formal calculational aspects of the model. In this work we will use most of the methodologies developed by Glaser, Pradhan and Scarf.

The definition of Green's functions of field theory which are vacuum expectation values of products of field operator do not directly require manipulations with asymptotic fields. Johnson (1961) doubted the existence of in and out fields for the model and obtained exact expressions for two and four particle Green's functions of the model by solving the differential equations for Green's functions. Using his result for the renormalized Green's function, was a simple matter for Wilson (1970) to show that the dimension of the field is given by

$$d = \tfrac{1}{2}+\frac{\lambda^2/4\pi^2}{1-\lambda^2/4\pi^2}. \qquad \dots (1)$$

The exact Green's functions were obtained by using the conservation of both vector and axial vector currents of the model. In realistic problems of field teory one gets an infinitely coupled set of equations for Green's functions. The conservation laws lead to two Ward identities for the vertex function which help in terminating the hierarchy of higher order Green's function equations. Johnson (1961) also proved that the charge densities do not commute and Glaser's results ceuld not be correct. The results of Scarf (1959) who followed Glassr would lead to $d = 1/2+g^2/8\pi^2$ instead of (1). Thus the correct and exact field theoretic solution of Thirring Model to the general equations of Lehmann, Symanzik and Zimmermann has not been obtained so far. This paper is aimed at providing this exact solution.

An elegant and very satisfactory solution to the Thirring model has been given by Klaiber (1968) using operator expansion method. Very recently solutions has also been obtained using the ideas of dilation generator and current commutators (Dell Antonido et al 1972). These methods do not solve the general equations of motion of fields directly. So the existence of Heisenberg in and out states are still kept in doubt. There is also the objections of Jackiw (1971) and others regarding the very existence of any anomaly in dimensions. The anomalous behaviour is attributed to the treatment of singular fuctions. The method of solution attempted by us clearly shows the manipulations involved for the singular integrals (section 4).

Besides, doubts have been raised regarding the use of Ward identities(Carruthers 1971). It is being contended that the anomalous results of field dimensions may be due to the use of these identities. We shall not use Ward identities and shall follow the normal method of constructing asymptotic wave functions and obtain the vacuum expectation values. We shall also obtain the same eq. (1) from the exact solution. The usefulness of the verification is that one can use the usual Feynman-Dyson perturbation theory to study the problem of anomalous dimensions provided the coupling strength is small.

The dimensionality as given by eq. (1) has one serious draw back in that it is infinity at $\lambda = 2\pi$. The dimensionality is derived from commutation relations and infinitely large departures are usually taken care of by introducing ideas of vertex renormalization. We shall show that within the conjectural framework of renormalizable field theory, it is possible to renormalize λ and obtain an equation

$$d = 1/2 + \frac{\left(\frac{1}{\pi}\sin^{-1}\frac{g_0}{2}\right)^2}{1-\left(\frac{1}{\pi}\sin^{-1}\frac{g_0}{2}\right)^2}. \qquad \dots \quad (2)$$

Even though this avoids the problem at infinity, it introduces ghost difficulties. This ghost difficulty is almost always encountered in quantum field theory.

The plan of the paper is as follows. In section 2 we write the solution in terms of Moller-like Matrix. The free and interaction Green's functions and their behaviour on scaling are also discussed. The normal ordering of the desired time ordered operators are carried out in section 3 in section 4 the renormalization constants, vacuum expectation values and dimensionality of field operator are calculated. The section 5 is devoted to a calculation of S matrix and a possible scheme of renormalization. In the concluding remarks of section 6 we discuss some implications of the dimensionality equations deduced by us.

2. Field Equations and Green's Function

The equation of motion for the field operators of the model is

$$-i\gamma_\mu\partial_\mu\Psi(x,t)+2g(\bar{\Psi}(x,t)\Psi(x,t))\Psi(x,t) = 0. \qquad \dots \quad (3)$$

In terms of Pauli-matrics, the γ matrics are given as

$$\gamma_1 = i\sigma_1;\ \gamma_2 = \beta = \sigma_2;\quad \gamma_5 = \gamma_1\gamma_2 = \sigma_3, \qquad \dots \quad (4)$$

and they satisfy the following anti-commutation relations

$$\{\gamma_\mu, \gamma_\mu\} = -2g_{\mu\nu}$$

$$g_{11} = -g_{22} = 1;\quad g_{12} = g_{21} = 0.$$

The field operators $\Psi\ (x,t)$ obey the equal-time anti-commutation relations

$$\{\Psi_\tau(x,t),\ \Psi_\tau{}^*(x,'\,t)\} = \delta_{\tau\tau'}\delta(x-x'), \tau, \tau' = 1, 2 \qquad \dots \quad (5)$$

and all other anti-commutators are zero. Introducing the hyperbolic co-ordinates

$$u = x+t;\ v = x-t$$

and using $$\{\Psi_\tau(x, t), \Psi_{\tau'}^*(x', t)\} = \delta_{\tau\tau'}\delta(x-x')$$

the field eq. (1) can be easily seperated in the variables u and v which provides the keys to the exact solutions,

$$\frac{\partial\Psi_1(u, v)}{\partial u} = ig(\Psi_2^*(u, v)\Psi_2(u, v))\Psi_1(u, v), \tag{6a}$$

$$\frac{\partial\Psi_2(u, v)}{\partial v} = -ig(\Psi_1^*(u, v)\Psi_1(u, v)\Psi_2(u, v). \tag{6b}$$

The free field equations are obtained by letting $g \to 0$. In this limit $\Psi_\tau \to \Phi_\tau$, free field operators which satisfy the following canonical anti-commutation relation

$$\{\Phi_1(v), \Phi_1^*(v')\} = \delta(v-v') \quad \ldots \quad (7a)$$

$$\{\Phi_2(u), \Phi_2^*(u'\} = \delta(u-u'). \quad \ldots \quad (7b)$$

All other anti-commutators vanish. In momentum representation,

$$\Phi_1(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dpC_1(p)e^{ipv} \quad \ldots \quad (8a)$$

$$\Phi_2(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dpC_2(p)e^{ipu}. \quad \ldots \quad (8b)$$

The non-vanishing anti-commutator of C_τ's is

$$\{C_\tau(p),\ C_{\tau'}^*(p')\} = \delta_{\tau\tau'}\delta(p-p'). \quad \ldots \quad (9)$$

In terms of creation and annihilation operators C_τ's are

$$C_{1,\,2}(p) = \theta_\pm(p)a(p)+\theta_\mp(p)b^*(-p) \quad \ldots \quad (10)$$

and the only nonvanishing anti-commutator is, as usual

$$\{a(p), a^*(p')\} = \{b(p), b^*(p')\} = \delta(p-p'). \quad \ldots \quad (11)$$

The solutions of the coupled field eq. (6) is correctly given by Pradhan (1958) treating them as a q-number system of field equations.

$$\Psi_1(u, v) = \Phi_1(v)U_1(u, -\infty), \quad \ldots \quad (12a)$$

$$\Psi_2(u, v) = \Phi_2(u)U_2(+\infty, v), \quad \ldots \quad (12b)$$

where

$$U_1(u, -\infty) = \left(e^{ig\int_{-\infty}^{u} \rho_2(u')du'}\right)_+, \quad \ldots \quad (13a)$$

and

$$U_2(+\infty, v) = \left(e^{ig\int_v^{+\infty} \rho_1(v')dv'}\right)_+, \tag{13b}$$

with

$$\rho_1(v') = \Psi_1{}^*(u, v')\Psi_1(u, v') = \Phi_1{}^*(v')\Phi_1(v'), \tag{14a}$$

$$\rho_2(u') = \Psi_2{}^*(u', v)\Psi_2(u', v) = \Phi_2{}^*(u')\Phi_2(u'). \tag{14b}$$

The symbol $(\ \ )_+$ denotes the Dyson's chronological ordering.

Glaser (1958) omitted the time ordering restriction contending that charge densities commute in a typical limiting proceedure. From the anti-commutators given above one can easily show that

$$<0\,|\,[\rho_2(u), \rho_2(u')]\,|\,0> = \frac{i}{2\pi}\,\delta'(u-u')$$

The right hand side is the Schwinger term. This result was derived first by Pradhan (1958) and subsequently confirmed by Johnson (1961).

Vacuum expectation values of the product of the operators at different space time points will be defined as the Green's function. With the help of these Green's functions we shall find the dimensions of the field as the scale transformations take a simple form for those functions. A typical Green's function is

$$\begin{aligned} G_{12}{}^+(u_1, v_1\,|\,u_2, v_2) &= <0\,|\,\Psi_1(u_1, v_1)\,\Psi_1{}^*(u_2, v_2)\,|\,0> \\ &= <0\,|\,\Psi_1(x_1, t_1)\Psi_1{}^*(x_2, t_2)\,|\,0> \\ &= G_{12}{}^0(v_1\,|\,v_2) < 0\,|\,U_1(u_1, u_2)\,|\,0>. \end{aligned} \tag{15}$$

$G_{12}{}^0$ is the free field Green's function whose value can be found simply writing Φ_1's in the momentum representation and evaluating the vacuum expectation value,

$$\begin{aligned} G_{12}{}^0(v_1\,|\,v_2) &= <0\,|\,\Phi_1(v_1)\Phi_1{}^*(v_2)\,|\,0> \\ &= \frac{1}{2\pi i}\,\frac{1}{v_2-v_1-i\epsilon}. \end{aligned} \tag{16}$$

The covariant form is easily deduced to be

$$G^0(x-y) = \frac{1}{2\pi i}\,\frac{\gamma.(x-y)}{(x-y)^2-i\epsilon}. \tag{17}$$

The behaviour of these Green's function under scale transormation is characterized as (Carruthers 1971)

$$x_\mu' = Sx_\mu$$

and field transforms as

$$\Psi(x, t) \to \Psi'(x', t') = S^d\Psi(sx, st)$$

and the field conjugate to ψ transforms as

$$\bar{\Psi}(x,t) \to \bar{\Psi}(x',t') = s^d \bar{\Psi}(sx', st')$$

where s is an arbitrary dimensionless real number and d is the dimension of the field Ψ in mass units or in inverse length units. Assuming that vacuum is invariant under scale transformations, the Green's function transforms under this transformation in the following manner

$$G_{12}{}^{+}(u_1, v_1 \mid u_2, v_2) = s^{2d} G_{12}{}^{+}(su_1, sv_1 \mid su_2, sv_2). \qquad \dots \quad (18)$$

This is an important relation and with the help of this the field dimension will be obtained. Using this relation, the free field dimension from eq. (17) is easily seen to be 1/2. For calculating the exact Green's function, it is easily deduced that

$$G_{12}{}^{+}(u_1, v_1 \mid u_2, v_2) = G_{12}{}^{0}(v_1 \mid v_2) < 0 \mid U_1(u_1, -\infty) U_1{}^{*}(u_2, -\infty) \mid 0 >$$
$$= G_{12}{}^{0}(v_1 \mid v_2) U_0$$

where

$$U_0 = <0 \mid U_1(u_1, u_2) \mid 0> = <0 \mid U_1 \mid 0>$$

and

$$U_1 = U_1(u_1, u_2) = U_1(u_1, -\infty) U_1{}^{*}(u_2, -\infty)$$
$$= \left(\exp\{ig \int_{-\infty}^{u_1} \rho_2(u')du'\}\right)_{+} \left(\exp\{-ig \int_{-\infty}^{u_2} \rho_2(u')du'\}\right)_{+}. \qquad (19)$$

3. Ordered Expressions

To determine vacuum expectation values of products of operators, first one has to order or anti-order the products. Let $N(U_1)$ and $\bar{N}(U_1)$ denote respectively the ordered and anti-ordered expressions of U_1. Then one can write

$$U_1(u_1, u_2) = <0 \mid U_1(u_1, u_2) \mid 0> N(U_1), \qquad (20)$$

or

$$U_1(u_1, u_2) = <\bar{0} \mid U_1(u_1, u_2) \mid \bar{0}> \bar{N}(U_1). \qquad (21)$$

Here $|\bar{0}>$ or $|CAV>$ is a state full of particles and antiparticles. After obtaining the correct expression for the ordered product of $U_1(u_1, \to\infty)$, Pradhan thought one has to antiorder $U_1(u_1, u_2)$ to calculate U_0 and calculated an expression like (21). But he omitted the normalizing factor $<\bar{0} \mid U_1 \mid \bar{0}>$ from his expression. This omission was pointed out by Scarf (1959). Actually one can show that the two factors $<0 \mid U_1 \mid 0>$ and $<\bar{0} \mid U_1 \mid \bar{0}>$ are related. It turns out that it is enough to obtain, correctly the ordered expression for U_1.

With Glaser (1958) and Scarf (1959), we begin by assuming that U_1 can be written as

$$U_1 = U_0 N \exp[\int dx \int dy H(x,y) \Phi_2{}^{*}(x) \Phi_2(y)], \qquad \dots \quad (22)$$

and the commutator,

$$[U_1, \Phi_2(x)] = \int dy E(x, y)[\Phi_2^+(y)U_1 + U_1\Phi_2^-(y)], \quad \dots (23)$$

where $\Phi_2^+(y)$ and $\Phi_2^-(y)$ are the positive and negative frequency parts of $\Phi_2(y)$. Now evaluating the quantity $<0|\{\Phi_2^*(x), [\Phi_2(y), U_1]\}|0>$ with the help of eqs. (22) and (23) separately and comparing one obtains $H(x, y) = -E(x, y)$.

The first step in the process of ordering is to calculate the commutator $[U_1, \Phi_2(x)]$ in a form as it is written in eq. (23). It has been noted that there are two different types of solutions because of the difference in the order of summing and integrating a series involving products of singular functions like Dirac's dalta function and θ-functions. These solutions will have bearing on some of our later discussions. Noting that

$$U_1(u_1, u_2) = U_1(u_1, -\infty)U_1^*(u_2, -\infty)$$

We shall write $T = U_1^{-1}(u_1, u_2)\Phi_2(x)\, U_1(u_1, u_2)$ using a well-known relation

$$U_1^{-1}(u, -\infty)\Phi_2(x)U_1(u, -\infty) = \Phi_2(x)\left[1 + \sum_{n=1}^{\infty}(ig)^n \times \int_{-\infty}^{u} du_1 \int_{-\infty}^{u_1} du_2 \dots \int_{-\infty}^{u_{n-1}} du_n \delta(u_n - x)\dots\delta(u_1 - x)\right]$$

In Dyson method, the n-th term of this sum is

$$t_n = (ig)^n \int_{-\infty}^{u} du_1 \dots \int_{-\infty}^{u_{n-1}} du_n[\delta(u_n - x)\dots\delta(u_1 - x)]\Phi_2(x)$$

$$= \frac{(ig)^n}{n!}\int_{-\infty}^{u} du_1 \dots \int_{-\infty}^{u} du_n P[\delta(u_n - x)\dots\delta(u_1 - x)]\Phi_2(x)$$

$$= \frac{(ig)^n}{n!}\left[\int_{-\infty}^{u} du'\delta(u' - x)\right]^n \Phi_2(x).$$

With Thirring the same term can be calculated as

$$t_n = (ig)^n \frac{1}{2^n}\theta(u - x)\Phi_2(x)$$

where $\theta(0) = \frac{1}{2}$ has been used for each integration. Summing up the series we obtain two expressions for T.

$$T = \left[\exp\left\{2i \tan^{-1}\frac{g}{2}\,(\theta(u_2 - x) - \theta(u_1 - x)\right\}\right]\Phi_2(x), \quad \dots (24a)$$

$$T = [\exp\{ig(\theta(u_2 - x) - \theta(u_1 - x)\}]\phi_2(x). \quad \dots (24b)$$

The first solution was considered by Thirring (1958) and the second by Glaser (1958). Pradhan (1958) calls the first solution as *standing wave* type and second as the *scattering type*. Pradhan considered only the second solution but we shall take both the solutions and write in one form with a 'dual' coupling constant λ so that taking different values both these solutions can be obtained. These anomalous results were first pointed out by Glaser (1958) and were subsequently discussed by Thirring (1958). Thirring noted that these may correspond in one case to $\lambda = g$ being restricted to have a magnitude less than π and where as for $\lambda = 2\tan^{-1} g/2$ it can have all possible values from $-\infty$ to $+\infty$. We can rewrite the eq. (24) as

$$[\Phi_2(x), U_1] = (1-e^{i\lambda})e^{-i\lambda\theta(u_2-u_1)}[\theta(u_2-x)-\theta(u_1-x)]U_1\Phi_2(x). \quad (24c)$$

Now we shall proceed to write this commutator in an ordered form. For simplicity we shall take $u_2 > u_1$ since

$$\Phi_2^+(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Phi_2(y)dy}{y-x-i\epsilon}$$

eq. (24c) can be written in the form

$$[\Phi_2^+(x), U_1] = (e^{-i\lambda}-1)\int_{u_1}^{u_2} \frac{U_1\Phi_2(y)dy}{y-x-i\epsilon}, \quad \ldots (25)$$

We write eq. (25) in the operator from i.e.,

$$[\Phi_2^+(x), U_1] = K^+U_1\Phi_2(x) \quad \ldots (26)$$

where K^+ denotes the operation

$$K^+f(x) = \int_{u_1}^{u_2} K^+(x, y)f(y)dy$$

where

$$K^+(x, y) = \frac{e^{-i\lambda}-1}{2\pi i}\,\frac{1}{y-x-i\epsilon}$$

we can write eq. (26) as

$$[\Phi_2^+(x), U_1] = K^+U_1\Phi_2^+(x)+K^+U_1\Phi_2^-(x)$$

or

$$[1+K^+]U_1\Phi_2^+(x) = \Phi_2^+(x)U_1-K^+U_1\Phi_2^-(x). \quad \ldots (27)$$

Further we write the above eq. as

$$[1+K^+]f(x) = g(x) \quad (28)$$

where $\quad f(x) = U_1\Phi_2^+(x)$

and $\quad g(x) = \Phi_2^+(x)U_1-K^+U_1\Phi_2^-(x).$

Eq. (28) is the integral equation that we have to solve. The solution of this integral equation is written in the form

$$f(x) = (1+E^+)g(x)$$

where
$$E^+g(x) = \int_{u_1} E^+(x, y)g(y)dy$$

and
$$[1+E^+] = [1+K^+]^{-1}$$

$E^+(x, y)$ is determined below.

The solution of the integral equation is obtained with the help of the jump relation (Glaser 1958, Scarf 1959). In terms of $p(x) = f(x)-g(x)$ we have

$$p(x_+)-p(x_-) = (1-e^{-i\lambda})[p(x_+)+g(x)],$$

$$x_\pm = x\pm i\epsilon; \quad \epsilon\rightarrow 0$$

Following standard method one writes

$$p(x) = S(x)Q(x)$$

Here

$$S(x) = \frac{1-e^{i\lambda}}{2\pi i}\int_{u_1}^{u_2} \frac{g(y)}{Q_+(y)}\frac{dy}{x-y}$$

and

$$Q(x) = \left(\frac{u_2-x}{u_1-x}\right)^{\lambda'/2\pi}$$

Hence the solution is

$$f(x) = g(x)+\int_{u_1}^{u_2} E^+(x, y)g(y)dy, \tag{29}$$

where

$$E^+(x, y) = \frac{1-e^{i\lambda}}{2\pi i}\left(\frac{u_2-y_+}{u_2-x_+}\right)^{\lambda'/2\pi}\left(\frac{y_+-u_1}{x_+-u_1}\right)^{\lambda'/2\pi}\frac{1}{x-y+ic}, \tag{30}$$

$$\lambda' = \lambda+2\pi n; \; n = 0, \pm 1, \pm 2, \ldots.$$

and the limit $u_2>x$, $y>u_1$ is considered.

The 'principal' solution is obtained for $n = 0$.

The ordered expression for $U_1(u_1, u_2)$ can be written with the help of eqs. (22), (23), (24c) and (29) as

$$U_1(u_1, u_2) = U_0N\exp[(e^{-i\lambda}-1)\times$$
$$\int_{u_1}^{u_2} dx\int_{u_1}^{u_2} dy\,\{\delta(x-y)+E^+(x, y)\}\phi_2^\gamma(x)\phi_2(y)]. \quad \ldots \tag{31}$$

This expression is identical to Scarfs' (1958). Similar expressions are obtained by Glaser (1958) and Pradhan (1958) in momentum representation. But all these authors adopted different methods in evaluating the vacuum expectation value $<0|U_1|0>$ which are not consistent. Glaser and later Scarf calculated this quantity by differentiating the expression

$$U_1(u_1, u_2) = \left(e^{-ig}\int_{-\infty}^{u_2}\rho_2(u')du'\right)_+\left(e^{ig}\int_{-\infty}^{u_1}\rho_2(u')du'\right)_+ \tag{32}$$

with respect to g, but but did not consider the time ordered expression for $U_1(u_1, u_2)$ and they write the above relation as

$$U_1(u_1, u_2) = e^{ig}\int_{u_2}^{u_1}\rho_2(u')du')$$

treating ρ's as c-numbers to obtain the differential equation,

$$\frac{\partial U_1}{\partial g} = ig\int_{u_2}^{u_1} du'\rho_2(u')U_1$$

which is incorrect for noncommuting ρ's as can be seen by directly differentiating eq. (32).

Pradhan's method in evaluating such vacuum expectation value is also not generally true. He obtained an antiordered expression for U_1, which in the present case can be written as

$$U_1(u_1, u_2) = C\bar{N}\exp[(e^{-i\lambda}-1)\times$$
$$\int_{u_1}^{u_2}dx\int_{u_1}^{u_2}dy\{\delta(x-y)+\bar{H}(x, y)\}\phi_2^*(x)\phi_2(y)] \tag{33}$$

$\bar{N}$ denotes the anti-ording i.e., all the annihilation operators are commuted to the left of creation operators. Pradhan took the normalization C the unknown constant which is yet to be determined (Scarf 1959), to be one. Such constant can be determined if we consider a state $|\text{CAV}>$ (Deo 1962) which is full of particles and antiparticles such that

$$a^*|\text{CAV}> = b^*|\text{CAV}> = 0; \quad <\text{CAV}|a = <\text{CAV}|b = 0.$$

So if we take the matrix element of the antiordered expression of $U_1(u_1, u_2)$ between the state $|\text{CAV}>$ then we have

$$C = <\text{CAV}|U_1|\text{CAV}>$$

So the anti-ordered expression of $U_1(u_1, u_2)$ is useless in evaluating the vacuum expectation value of this operator because for this we have to find another unknown constant which is equally difficult and is actually related to U_0.

4. Vacuum Expectation Values

The defining differential equation for $U_1(u_1, u_2)$ is to be used for fixing the unknown constant. Differentiating the expression (32) with respect to u_2 we have

$$\frac{\partial U_1}{\partial u_2} = -ig : \rho_2(u_2) : U_1. \tag{34}$$

If we take the vacuum expectation value of the above equation we have

$$U_0' = -ig < 0 | \ : \rho_2(u_2) : U_1 | 0 >$$

where

$$U_0' = \frac{\partial U_0}{\partial u_2}$$

U_0 is already defined. We write r.h. side as

$$-ig\Sigma <0| : \rho_2(u_2) : \ |n><n| U_1 |0> \qquad \ldots \quad (35)$$

where $|n>$ is a complete set of orthonormal states. Examining carefully the last equation one can see that only $n = 2$ state gives the non-vanishing contribution. Because for $n>2$ the matrix element $<0| : \rho_2(u_2) : |n>$ vanishes, as $\rho_2(u_2)$ contains only bilinear operators that can only annihilate or create a pair of particle and antiparticle.

In eq. (35) we shall put ordered expression for $U_1(u_1, u_2)$ and only second term gives the non-vanishing contribution and we can write eq. (35) as

$$U_0' = -igU_0(e^{-i\lambda}-1) \int_{u_1}^{u_2} dx \int_{u_1}^{u_2} dy[\delta(x-y)+E^+(x, y)] \times$$
$$<0| : \rho_2(u_2) : : \Phi_2^*(x)\Phi_2(y) : |0> \qquad \ldots \quad (36)$$

the quantity $<0| : \rho_2(u_2) : : \Phi_2(x)\Phi_2(y) : |0>$ can be easily evaluated expanding Φ_2's in terms of creation and annihilation operators. This value comes out to be

$$-\frac{1}{4\pi^2} \frac{1}{(x-u_2+i\epsilon)(y-u_2+i\epsilon)}$$

Eq. (36) can be written as

$$U_0' = \frac{ig}{4\pi^2} U_0 \int_{u_1}^{u_2} dx \int_{u_1}^{u_2} dy \frac{[\delta(x-y)+E^+(x, y)]}{(x-u_2+i\epsilon)(y-u_2+i\epsilon)} e^{-i\lambda}-1. \qquad \ldots \quad (37)$$

This is the expression from which we shall obtain U_0 after a subsequent integration. To perform the integrals of this equation we shall repeatedly make use of the relations obtained from the differential equation or its solution namely

$$[1+E^+]K^+ = -E^+$$

writing

$$\frac{e^{-i\lambda}-1}{2\pi i}\ \frac{1}{u_2-y-i\epsilon} = K^+(y, u_2). \tag{38}$$

With the help of eq. (38) we can write eq. (37) as

$$U_0' = \frac{g}{2\pi} U_0 \int_{u_1}^{u_2} dx \frac{E^+(x, u_2)}{u_2-x-i\epsilon}. \tag{39}$$

To proceed further we introduce another variable by the use of a Dirac delta function, i.e., we write U_0' as

$$U_0' = \frac{g}{2\pi} U_0 \int_{u_1}^{u_2} dx \int_{u_1}^{v_2} dt \frac{E^+(x, t)}{t-x-i\epsilon}\ \delta(u_2, -t), \tag{40}$$

the integrand in the above equation can be written in the form $K^+(t, x)\ E^+(x, t)$ by simply writing $(t-x-i\epsilon)^{-1} = (t-x+i\epsilon)^{-1}+i\pi\delta(x-t)$. Again using the relation $E^+K^+ = -E^+-K^+$, eq. (40) can be written in the following form

$$U_0' = -\frac{g}{2\pi} U_0 \int_{u_1}^{u_2} dx \int_{u_1}^{u_2}\ dt \times$$

$$\left[1-\left(\frac{u_2-t_+}{t_+-u_1}\right)^{\lambda'/2\pi}\left(\frac{u_2-x_+}{x_+-u_1}\right)^{\lambda'/2\pi}\right]\ \frac{\delta(x-t)\delta(u_2-t)}{x-t+i\epsilon}. \tag{41}$$

It is clear that perturbation expansion will lead to a term proportional to $g\lambda$ in the second order and in stationary type of solution to $2g \tan^{-1} g/2$.

It turns out the integrand is in an indeterminate form if we do x integration. Scarf (1959) also comes across this expression in a similar context. Analysing the integrand carefully one finds that the correct value for the integrand can be obtained only if we expand the term in the bracket into a power series by exponentiating the product factors. One assumes that the expansion in powers of $\lambda'/2\pi$ converges. Now

$$U_0' = -\frac{g}{2\pi} U_0 \sum_{n=1}^{\infty} \left(\frac{\lambda'}{2\pi}\right)^n \frac{1}{n!} I_n$$

$$I_n = \int_{u_1}^{u_2} dx \int_{u_1}^{u_2} dt\ L^n\ \frac{\delta(x-t)\delta(u_2-t)}{t-x-i\epsilon}$$

where

$$L^n = \left[\log \frac{u_2-x_+}{x_+-u_1} - \log \frac{u_2-t_+}{t_+-u_1}\right]^n$$

$(t-x-i\epsilon)^{-1}\delta(x-t)$ equals a $\delta'(x-t)$ and $[\delta(x-t)]^2$. It is seen that due to symmetry of integration for x and t variables for odd n, L^n changes sign on interchange of these variables, the odd n terms contain the $\delta'(x-t)$ part and the even terms $[\delta(x-t)]^2$ part. However, the smallest even n being 2, near $x = t$, L^2 tends to $(x-t)^2$ and using the fact that $[(x-t)\delta(x-t)]^2 = 0$, we can drop this singular series from our consideration. To integrate the $\delta'(x-t)$ we see that when taking the limit by using L. Hopital's rule, we find the same indeterminate form recurring as is explicitly shown below. First time we obtain

$$nL^{n-1}\left(\frac{1}{u_2-x_+}+\frac{1}{x_+-u_1}\right)\delta(x-t)\delta(u_2-t)$$

and write it as

$$nL^{n-}\frac{(u_2-u_1)}{(u_2-x_+)(x_+-u_1)}\quad\delta(x-t)\delta(u_2-t)$$

$$= nL^{n-1}\frac{1}{t-x-i\epsilon}\delta(x-t)\delta(u_2-t). \qquad \ldots (42)$$

Eq. (42) is again in the indeterminate form. We have to use L' Hospital's rule again. Repeating this n times we get the final expression that looks like

$$n(n-1)(n-2)\ldots 3.2.1\left[\frac{1}{u_2-x_+}+\frac{1}{x_+-u_1}\right]. \qquad (43)$$

Putting these expression in eq. (41) and summing up we obtain

$$U_0' = -\frac{g}{4\pi}\;\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\;U_0\int_{u_1}^{u_2}dx\int_{u_1}^{u_2}dt\,\times$$

$$\left[\frac{1}{u_2-x_+}+\frac{1}{x_+-u_1}\right]\delta(x-t)\delta(u_2-t). \qquad (44)$$

Now we can perform integration over x and the result is

$$U_0' = -\frac{g}{2\pi}\;\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\;U_0\int_{u_1}^{u_2}dt\times\left[\frac{1}{u_2-t_+}+\frac{1}{t_+-u_1}\right]\delta(u_2-t). \qquad (45)$$

The integrand in the above integral is not finite as expected. We shall rewrite the term in the bracket as

$$-\int_{u_1}^{u_2}du''\,\frac{1}{(u''-t_+)^2}$$

and obtain after integrating over t,

$$U_0' = \frac{g}{2\pi}\;\frac{\lambda'}{2\pi}\;\frac{1}{1-\lambda'^2/4\pi^2}U_0\int_{u_1}^{u_2}du''\left[\frac{1}{u''-u_{2+}}\right]^2. \qquad \ldots (46)$$

The above expression can now be integrated to give the result in a desired form :

$$U_0 = \exp\left[\frac{g}{2\pi}\,\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\int_{u_1}^{u_2} du' \int_{u_1}^{u'} du''\,\frac{1}{(u''-u_+')^2}\right]. \quad \ldots \quad (47)$$

This expression is different from that of Scarf (1959). The correctness of eq. (47) can be checked by evaluating U_0 from perturbation theory. The second order results are exactly same, if we put $\lambda' = g$ or $\lambda' = 2\tan^{-1} g/2$. Actually in the first order one obtains the term proportional to $g \times 2\tan^{-1} g/2 = \lambda \times 2\tan^{-1}\lambda/2$ which was also obtained by Thirring (1959) while discussing Glaser's results and he wrote about this as being the geometric mean.

At this point, it is also worth noting that, by repeating, the whole calculation one will obtain

$$<0\,|\,U_2(v_1, v_2)\,|\,0> = \exp\left[\frac{g}{2\pi}\,\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\int_{v_1}^{v_2} dv' \int_{v_1}^{v'} dv''\left\{\frac{1}{v''-v_+'}\right\}^2\right].$$

and by a linear mapping from the u space to v space, one can show that

$$<0\,|\,U_2(v_1, v_2)\,|\,0> = <0\,|\,U_1(u_1, u_2)\,|\,0>$$

$$= (<0\,|\,U_1(u_1, u_0)\,|\,0><0\,|\,U_2(v_1, v_2)\,|\,0>)^{1/2}$$

This result is useful in obtaining the covariant form of the Green's functions. We shall proceed in the hyperbolic variables for the present and write the renormalized vacuum expectation value as,

$$<0\,|\,U_1(u_1, u_2)\,|\,0>^{\mathrm{R}} = Z^{-\frac{1}{2}}Z^{*-\frac{1}{2}}<0\,|\,U_1(u_1, u_2)\,|\,0> \quad \ldots \quad (48)$$

where $\qquad Z^{*\frac{1}{2}} = <0\,|\,U_1^*(u_2, -\infty)\,|\,0>$

and $\qquad Z^{\frac{1}{2}} = <0\,|\,U_1(u_1, -\infty)\,|\,0>$

are the field renormalization constants. By taking suitable limits and complex conjugation one can determine these wave function renormalization constants. We can write these constants as

$$Z^{*\frac{1}{2}} = \exp\left[\frac{g}{2\pi}\,\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\int_{-\infty}^{u_2} du' \int_{-\infty}^{u'} du''\,\frac{1}{(u''-u'_+)^2}\right]. \quad \ldots \quad (49)$$

and

$$Z^{\frac{1}{2}} = \exp\left[\frac{g}{2\pi}\,\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\int_{-\infty}^{u_1} du' \int_{-\infty}^{u'} du''\,\frac{1}{(u''-u_-')^2}\right]. \quad \ldots \quad (50)$$

Now with the help of eqs. (49) and (50) we can write eq. (48) as

$$<0\,|\,U_1(u_1, u_2)\,|\,0>^{\mathrm{R}} = exp\left[-\frac{g}{2\pi}\,\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\int_{-\infty}^{u_2} du' \int_{-\infty}^{u_1}\frac{du''}{(u''-u'_+)^2}\right] \quad \ldots \quad (51)$$

our aim is now to obtain a covariant form of the Green's function. As it can be seen that the expression (51) contains only $u = x+t$ and the integral

$$\int^{u} du' \int_{-\infty}^{u'} \frac{du''}{(u''-u_{+}')^2}$$

is not covariant. In the hyperbolic co-ordinates, the above calculations have left $v_1 = x_1-t_1$ and $v_2 = x_2-t_2$ unspecified. The substitution $y' = u''-u_1$ and $x' = u'-u_1$ followed by

$$x = (v_2-v_1)x'$$

$$y = (v_2-v_1)y'$$

brings the integral to the desired covariant form namely

$$\int^{(v_2-v_1)(u_2-u_1)} dx \int_{-\infty}^{0} \frac{dy}{(y-x-i\delta)^2} \int^{(v_2-v_1)(u_2-u_1} \frac{dx}{x+i\delta}.$$

Since

$$(v_2-v_1)(u_2-u_1) = (x_{2\mu}-x_{1\mu})^2, \tag{52}$$

a covariant cut-off can be used to integrate the above expression and the result is

$$\log \frac{(x_{2\mu}-x_{1\mu})^2}{x_0^2}$$

finally the expression of the vacuum expectation value is

$$U_0 = \exp\left[-\frac{g}{2\pi}\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\log\left(\frac{x_{2\mu}-x_{1\mu}}{x_0}\right)^2\right]. \tag{53}$$

and the Green's function is

$$G(x-y) = G^0(x-y)\, exp\left[-\frac{g}{2\pi}\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2}\log\left(\frac{x_{2\mu}-x_{1\mu}}{x_0}\right)^2\right]. \tag{54}$$

This is the covariant form of the Green's function identical to Johnson's result. Johnson Green's function is obtained if we take $\lambda' = g$ and for this choice the expansion is not an analytic function of g and has an essential singularity at $g^2 = 4\pi^2$. Thus we see that normal methods of field theory also lead to the same result as the use of Ward identites, as it should be. Using the above result the dimension from eq. (18) of section (2) is given by

$$d = \tfrac{1}{2}+\frac{g}{2\pi}\frac{\lambda'/2\pi}{1-\lambda'^2/4\pi^2} \tag{55}$$

It is to be noted that there is a singularity in the Green's function for $\lambda' = 2\pi$ which makes the dimensionality go to $\pm\infty$ round this value.

5. S-Matrix and Renormalization

The solutions of the field equations can now be written down explicitly and the matrix element of Heisenberg wave function for production of arbitrary number of particles can be obtained exactly. With

$$\Psi_1(u, v) = \Phi_1(v)U_1(u, -\infty)$$

$$\Psi_2(u, v) = \Phi_2(u)U_2(+\infty, v)$$

the normal ordered expression for U_1 can be gotten from our eq. (31)

$$U_1^*(u, -\infty) = <0\,|\,U_1^*(u, -\infty)\,|\,0> N\, exp\Big[(e^{-i\lambda}-1)\int_{\infty}^{u} dx \int_{-\infty}^{u} dy \times$$

$$\Big\{\delta(x-y) + \frac{1-e^{i\lambda}}{2\pi i}\left(\frac{u-y_+}{u-x_+}\right)^{-\lambda'/2\pi}\frac{1}{x-y+i\epsilon}\Big\}\Phi_2^*(x)\Phi_2(y)\Big]. \qquad (56)$$

A similar expression for U_2 can be obtained. Since the limit $t\to\infty$ implies $u = x+t\to\infty$ and $v\to-\infty$ the Heisenberg *out* fields are products of free fields and the $U_T(+\infty, -\infty)$. In this limit the above expression simplified considerably and going over to the momentum representation

$$\Psi_1^*{}_{out}(u, v) = \Phi_1(v)U_1(-\infty, +\infty)$$

where

$$U_1(-\infty, +\infty) = <0\,|\,U_1(-\infty, +\infty)\,|\,0> N\, exp\Big[\int_{\infty}^{+\infty} dp \times$$

$$\theta(-p)\{(e^{-i\lambda}-1)a^*(p)a(p)+(e^{i\lambda}-1)b^*(p)b(p)\}\Big]$$

$$= <0\,|\,U_1(-\infty, +\infty)\,|\,0> V_a V_b. \qquad (57)$$

V_a and V_b are the normal order exponential containing a's and b's respectively.

To simplify further, we drop writing the renormalization constants $<0\,|\,U_1(-\infty, +\infty)\,|\,0>$ assuming to be working with renormalized wave functions. We start by *deordering* V_a.

$$V_a = :\exp\int_{-\infty}^{\infty} dp(e^{-i\lambda}-1)a^*(p)a(p)\theta(-p):$$

$$\frac{\partial V_a}{\partial a^*(q)} = (e^{-i\lambda}-1)V_a a(q)\theta(-q). \qquad \dots (58)$$

The symbolic operator differential is shown by Schwinger (1954) to be equal to the commutator $[a(q), V_a]$. With some algebra we obtain

$$V_a^{-1}a(q)V_a = e^{-i\lambda\theta(-q)}a(q)$$

which gives the *deordered* product for V_a

$$V_a = exp[-i\lambda\int\theta(-p)a^*(p)a(p)dp]. \qquad \dots (59)$$

The product $V_a V_b$ can be recombined to give $V_a V_b = e^{-i\lambda Q_2}$ where Q_2 is one of the constants of the model $Q_{1,2} = \int_{-\infty}^{+\infty} :\rho_{1,2}(x): dx$ Carrying out a similar procedure for $\Psi_2(u, v)$ one obtains the results

$$\Psi_{1\,out}(u, v) = \Phi_1(v) e^{i\lambda Q_2}, \quad \ldots \quad (60a)$$

$$\Psi_{2\,out}(u, v) = \Phi_2(u) e^{i\lambda Q_1}, \quad \ldots \quad (60b)$$

These simplifying features had led Glaser to treat the charge density operators as c-numbers. By definition of the S-matrix

$$\Psi_{\tau\,out}(x, t) = S^{-1}\Phi_\tau(x, t)S. \quad \ldots \quad (61)$$

and since further

$$[Q_\tau, \Phi_\tau(x, t)] = -\delta_{\tau\tau'}\Phi_{\tau'}(x, t) \quad \ldots \quad (62)$$

the S matrix takes the simple form

$$S = \left(e^{i\lambda \int_{-}^{+\infty} dx \int_{-\infty}^{+\infty} dt (\Psi(x,t)\Psi(x,t))^2} \right)_+$$

$$= e^{i\lambda \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv\, \rho_2(u)\rho_1(v}$$

$$= e^{i\lambda Q_1 Q_2}\,. \quad (63)$$

Glaser had deduced and Pradhan noted that $S = e^{igQ_1Q_2}$. It is also seen that Thirring's original solution would lead to

$$S = e^{i(2\tan^{-1} g/2)Q_1Q_2}. \quad (64)$$

The scattering matrix gives a phase change as is expected of a one space dimension problem. The ambiguities in the solutions reflected by two possible values of λ are also known in field theory (Karplus & Kroll 1950). In electrodynamic calculations they correspond to finite renormalization. Therefore it is possible here also that we may have a situation where renormalization of coupling strength is necessary, eventhough, the finiteness of the ratio of the three particles to one particle matrix elements led Thirring to believe that here is no coupling constant renormalization. Our *scattering solution* could be renormalized and this has also been pointed out by Pradhan (1958).

The renormalization procedure can be carried out by defining the observable renormalised coupling constant g_0 in such a way that the observable transmission coefficient or phase shift at vanishingly small momenta is equal to its Born approximation value in bare coupling constant. From the S matrix we easily deduce that

$$g_0 = 2 \sin \frac{g}{2} \quad (65a)$$

or

$$g = 2 \sin^{-1} \frac{g_0}{2}. \tag{65b}$$

For this case then there is an upper limit to the observed coupling constant namely $g_{0max} = 2$. This implies that the usual Kallen-Pauli Ghost difficulty is likely to appear in this model also for $g_0 > 2$. The dimension of the field will be

$$d = \tfrac{1}{2} + \frac{\left(\frac{1}{\pi} \sin^{-1} \frac{g_0}{2}\right)^2}{1 - \left(\frac{1}{\pi} \sin^{-1} \frac{g_0}{2}\right)^2}, \tag{61}$$

in terms of renormalized coupling constant d will become complex if $g_0 > 2$.

6. Conclusion

To summarise, in this paper we have shown that starting with the solutions of field equations, the exact Green's function can be obtained. Because of different approaches in handling the product of singular functions, there are two different expressions for the commutator $[U_1, \Phi_2(x)]$ (eq. 24c). We found that the Green's function derived by us are identical with that of Johnson (1961) for the scattering type of solution. Johnson had obtained the Green's function with the help of Ward identities and the interaction he considered was current-current. Field dimension computed by us in the present treatment are not different from that of Wilson in terms of bare coupling constant since he used Johnson's results in deducing the same. It is seen that the dimension are anomalous in both the cases. It is also pointed out that dimension of the field becomes infinity in the limit $\lambda' = 2\pi$. This point forces us to think that there must be some renormalization procedure so that the dimension may stay finite, since infinite field dimension does not make sense as the dimension of the Lagrangian is fixed. So one may use the renormalized coupling constant in the Thirring model. The renormalized coupling constant as defined by Pradhan (1958) and Deo (1962) is used in place of unrenormalized ones and it is found that the real dimension are limited between 1/2 and 5/6. It is also observed that when g_0 exceeds the value 2 the dimension of the field becomes complex.

We have shown that the exactly soluble Thirring model shows all the features of Quantum Field theory. As a consequence the Feynman-Dyson perturbation approach is found to be reliable for small coupling strengths and for studying small departures of dimensionality of fields.

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